WISKUNDIGE LOGICA (2016-2017)
HOMEWORK 2

- Deadline: Monday, March 14 — at the **beginning** of class.
- Homework can also be submitted electronically (in a single pdf-file!) to Jolien Oomens).
- Grading is from 0 to 100 points.
- Success!

(1) (20pt) Let $\mathfrak{A}$ and $\mathfrak{B}$ be S-structures, and let $\pi : \mathfrak{A} \cong \mathfrak{B}$. Let β be an assignment on $\mathfrak{A}$ and assume that $\beta^\pi = \pi \circ \beta$. Show that for every S-term t we have $\pi(\mathfrak{I}(t)) = \mathfrak{I}^\pi(t)$, where $\mathfrak{I} = (\mathfrak{A}, \beta)$ and $\mathfrak{I}^\pi = (\mathfrak{B}, \beta^\pi)$.

(2) (20pt)
  (a) Write a formula φ_n for $n \in \mathbb{N}$ and $n > 0$ such that a structure $\mathfrak{A}$ is a model of φ_n iff $\mathfrak{A}$ contains at least (!) n points.
  (b) Write a formula ψ_n for $n \in \mathbb{N}$ and $n > 0$ such that a structure $\mathfrak{A}$ is a model of ψ_n iff $\mathfrak{A}$ contains at most (!) n points.
  (c) Write a formula χ_n for $n \in \mathbb{N}$ and $n > 0$ such that a structure $\mathfrak{A}$ is a model of χ_n iff $\mathfrak{A}$ contains exactly (!) n points.

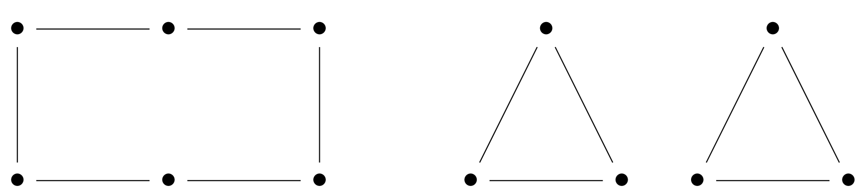
(3) (20pt) A set Φ of sentences is called *independent* if there is no $\varphi \in \Phi$ such that $\Phi \setminus \{\varphi\} \models \varphi$. Show that the set Φ_{eq} of axioms (reflexivity, symmetry, transitivity) for equivalence relations is independent.

(4) (20pt) Let $S = \{E\}$ with E a binary relation symbol. An S-Structure $\mathfrak{G} = (G, E^{\mathfrak{G}})$ is called a graph if it is a model of

$$\Phi_{graph} = \{\forall v_0 \neg E v_0 v_0, \forall v_0 \forall v_1 (E v_0 v_1 \leftrightarrow E v_1 v_0)\}$$

(Intuitively: A graph consists of a set G; two points $a \neq b$ are related iff $(a, b) \in E^{\mathfrak{G}}$).

  (a) Are the following two graphs isomorphic ?



$\mathfrak{G}_1$ $\mathfrak{G}_2$

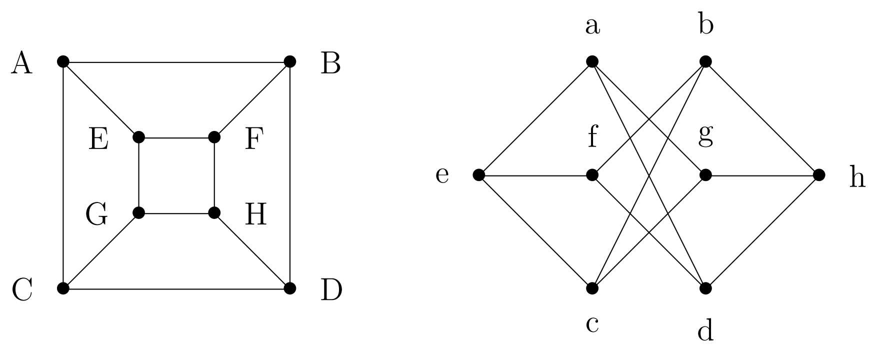
  (b) Does $\mathfrak{G}_i \models \varphi_j$, for $i = 1, 2$ and $j = 1, 2, 3$?

(i) $\varphi_1 = \exists v_0 \exists v_1 \exists v_2((Ev_0v_1 \wedge Ev_1v_2) \wedge Ev_2v_0)$

(ii) $\varphi_2 = \forall v_0 \forall v_1(\neg Ev_0v_1 \to \exists v_2(Ev_0v_2 \wedge Ev_2v_1))$

(iii) $\varphi_3 = \forall v_0 \exists v_1 \exists v_2 \exists v_3 \exists v_4((Ev_0v_1 \wedge Ev_0v_2) \wedge (Ev_0v_3 \wedge Ev_0v_4))$

(c) Show that the following two graphs are isomorphic:



(5) (20pt) Let S be a finite symbol set.

(a) Let $\mathfrak{A}$ be a finite S-structure. Show that there is an S-sentence $\varphi_{\mathfrak{A}}$ the models of which are precisely the S-structures isomorphic to $\mathfrak{A}$.

(b) We write $\mathfrak{A} \equiv \mathfrak{B}$ if $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the same S-sentences. If $\mathfrak{A}$ is finite, does $\mathfrak{A} \equiv \mathfrak{B}$ imply $\mathfrak{A} \cong \mathfrak{B}$? Justify your solution.