Zomercursus Wiskunde B

Week 3, les 4 (extra)

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Rijen en limieten

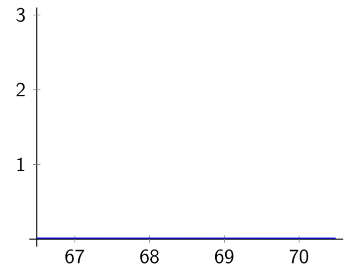
Bekijk de *rij* $a_n = 1/n$:

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots$$

We zijn nu geïnteresseerd in wat er gebeurt met a_n als n heel groot wordt. In dit geval wordt a_n steeds kleiner: de rij *convergeert* naar 0.

Notatie

$$\lim_{n\to\infty} \frac{1}{n} = 0.$$



Limieten

We zagen $\lim_{n\to\infty} \frac{1}{n} = 0$. Zo ook:

$$\lim_{n\to\infty} \frac{5}{n} = 0$$

$$\lim_{n\to\infty} \left(2 + \frac{1}{n}\right) = 2$$

$$\lim_{n\to\infty} \frac{1}{n+3} = 0$$

$$\lim_{n\to\infty} \frac{3}{n^2} = 0$$

$$\lim_{n\to\infty} \frac{\frac{1}{n} + 1}{\frac{1}{n^2} + 2} = \frac{1}{2}$$

$$\lim_{n\to\infty} \cos\left(\frac{1}{n}\right) = \cos(0) = 1$$

$$\lim_{n\to\infty} e^{3+\frac{2}{n^3}} = e^3$$

We gebruiken hierbij de regels:

$$\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} (a_n \cdot b_n) = \left(\lim_{n\to\infty} a_n\right) \cdot \left(\lim_{n\to\infty} b_n\right)$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} \qquad \left(\lim_{n\to\infty} b_n \neq 0\right)$$

$$\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right)$$

Quotiënten van veeltermen

Hoe berekenen we de limiet van een rij als $a_n = \frac{n^2+3n-2}{n^2-5n+1}$?
Truc: deel boven en onder door de hoogste macht van n:

$$\lim_{n\to\infty} \frac{n^2+3n-2}{n^2-5n+1} = \lim_{n\to\infty} \frac{\frac{n^2}{n^2} + \frac{3n}{n^2} - \frac{2}{n^2}}{\frac{n^2}{n^2} - \frac{5n}{n^2} + \frac{1}{n^2}} = \lim_{n\to\infty} \frac{1 + \frac{3}{n} - \frac{2}{n^2}}{1 - \frac{5}{n} + \frac{1}{n^2}}$$

$$= \frac{1+0-0}{1-0+0} = \frac{1}{1} = 1.$$

Zo ook:

$$\lim_{n\to\infty} \frac{2n^2-3}{n^3+5n} = \lim_{n\to\infty} \frac{\frac{2n^2}{n^3} - \frac{3}{n^3}}{\frac{n^3}{n^3} + \frac{5n}{n^3}} = \lim_{n\to\infty} \frac{\frac{2}{n} - \frac{3}{n^3}}{1 + \frac{5}{n^2}} = \frac{0}{1} = 0.$$

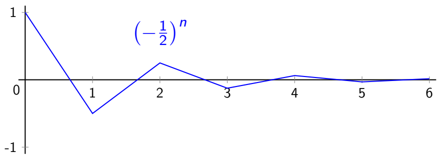
Limieten van machten

We hebben

$$\lim_{n\to\infty} \frac{1}{n^k} = 0 \qquad \text{als } k > 0$$

$$\lim_{n\to\infty} a^n = \begin{cases} \infty & \text{als } a > 1 \\ 0 & \text{als } -1 < a < 1 \end{cases}$$

2^n	2	4	8	16	32	64	$\cdots$
$\left(\frac{1}{2}\right)^n$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\cdots$
$(-2)^n$	-2	4	-8	16	-32	64	$\cdots$
$\left(-\frac{1}{2}\right)^n$	$-\frac{1}{2}$	$\frac{1}{4}$	$-\frac{1}{8}$	$\frac{1}{16}$	$-\frac{1}{32}$	$\frac{1}{64}$	$\cdots$



Limieten van machten

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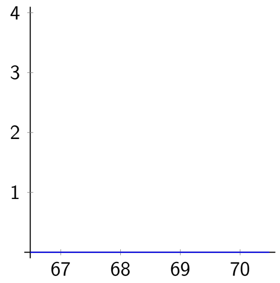
$$\lim_{n\to\infty} a^n = \begin{cases} \infty & \text{als } a > 1 \\ 0 & \text{als } -1 < a < 1 \end{cases}$$

$$\lim_{n\to\infty} \frac{n^k}{a^n} = 0 \qquad \text{als } a > 1 \text{ of } a < -1$$

$$\lim_{n\to\infty} n^k a^n = 0 \qquad \text{als } -1 < a < 1$$



n	1	2	3	4	5	6	7	8	9	10	11
n^3	1	8	27	64	125	216	343	512	728	1000	1331
2^n	2	4	8	16	32	64	128	256	512	1024	2048

Voorbeelden

We bekijken een aantal limieten:

$$\lim_{n\to\infty} \frac{1}{n^k} = 0, \quad k > 0$$

$$\lim_{n\to\infty} a^n = 0, \quad |a| < 1$$

$$\lim_{n\to\infty} \frac{n^k}{a^n} = 0, \quad |a| > 1$$

$$\lim_{n\to\infty} \frac{n^2+1}{2^n} = \lim_{n\to\infty} \left(\frac{n^2}{2^n} + \frac{1}{2^n}\right) = 0$$

$$\lim_{n\to\infty} \frac{3^n}{2^{2n}} = \lim_{n\to\infty} \frac{3^n}{4^n} = \lim_{n\to\infty} \left(\frac{3}{4}\right)^n = 0$$

$$\lim_{n\to\infty} \frac{n}{2^{n-3}} = \lim_{n\to\infty} \frac{n}{2^n 2^{-3}} = \frac{1}{2^{-3}} \lim_{n\to\infty} \frac{n}{2^n} = 0$$

$$\lim_{n\to\infty} \frac{2^n - n}{3^n + 1} = \lim_{n\to\infty} \frac{\frac{2^n}{3^n} - \frac{n}{3^n}}{\frac{3^n}{3^n} + \frac{1}{3^n}} = \lim_{n\to\infty} \frac{\left(\frac{2}{3}\right)^n - \frac{n}{3^n}}{1 + \frac{1}{3^n}} = \frac{0-0}{1+0} = 0$$

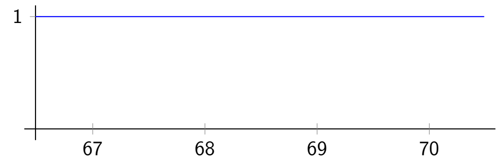
$$\lim_{n\to\infty} \frac{n^3 - 2^{-n}}{5n^3 + n} = \lim_{n\to\infty} \frac{n^3 - \left(\frac{1}{2}\right)^n}{5n^3 + n} = \lim_{n\to\infty} \frac{\frac{n^3}{n^3} - \left(\frac{1}{2}\right)^n / n^3}{\frac{5n^3}{n^3} + \frac{n}{n^3}}$$

$$= \lim_{n\to\infty} \frac{1 - \left(\frac{1}{2}\right)^n / n^3}{5 + \frac{1}{n^2}} = \frac{1-0}{5+0} = \frac{1}{5}$$

Bijzondere limieten

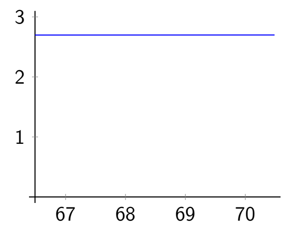
Twee belangrijke limieten:

$$\lim_{n\to\infty} n \sin\left(\frac{1}{n}\right) = 1$$



$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$$

$$\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$$



Voorbeelden

We bekijken een aantal limieten:

$$\lim_{n\to\infty}\left(1-\frac{2}{n}\right)^n = \lim_{n\to\infty}\left(1+\frac{-2}{n}\right)^n = e^{-2}$$

$$\lim_{n\to\infty}\left(1+\frac{2}{3n}\right)^n = \lim_{n\to\infty}\left(1+\frac{2/3}{n}\right)^n = e^{\frac{2}{3}}$$

$$\lim_{n\to\infty}\frac{n^2\sin\left(\frac{1}{n}\right)}{n+3} = \lim_{n\to\infty}\frac{n}{n+3}\cdot n\sin\left(\frac{1}{n}\right)$$

$$= \lim_{n\to\infty}\frac{n}{n+3} = \lim_{n\to\infty}\frac{1}{1+\frac{3}{n}} = 1$$

$$\lim_{n\to\infty} n\ln\left(1+\frac{2}{n}\right) = \lim_{n\to\infty}\ln\left(1+\frac{2}{n}\right)^n = \ln e^2 = 2$$

$$\lim_{n\to\infty}\frac{1}{n^k} = 0, \quad k>0$$

$$\lim_{n\to\infty} a^n = 0, \quad |a|<1$$

$$\lim_{n\to\infty}\frac{n^k}{a^n} = 0, \quad |a|>1$$

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$$

$$\lim_{n\to\infty} n\sin\left(\frac{1}{n}\right) = 1$$

Opgaven en indeling

Opgaven

8.16 abcf, 8.17 aef, 8.18 bce, 8.20 ab, 8.21 abde, 8.22 ab, 8.23 ae, 8.24 ab, extra.

Antwoorden van de opgaven staan achterin, uitwerkingen van de extra opgaven op http://www.bliggy.net/cursusB.html.

Groepen

De indeling is op basis van je achternaam:

- A t/m D: zaal A1.14 (Gideon Jager)
- E t/m Kuhl: zaal A1.30 (Jeroen Eijkens)
- Kuhlhan t/m Seydel: zaal D1.114 (Sebastian Zur)
- Simsir t/m Z: zaal D1.116 (Thijs Benjamins)