Beyond holomorphic functions

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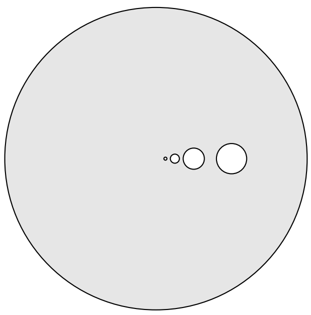

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Abstract

In this thesis, we continue the work of Gang Huang in [2] and study a topology on (subsets of) the complex plane that is distinct from the usual Euclidean and fine topologies. A notion of holomorphy with respect to this topology is introduced and standard results from complex analysis are derived. In particular, we consider rational approximation, the maximum principle and unicity of power series. For the latter, we see that we can construct a very similar topology for which unicity does not hold.

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Introduction

Complex analysis is concerned with studying holomorphic functions, i.e. functions that are differentiable in the complex sense. It is well known that this is a much stronger property than ordinary (real) differentiability. Indeed, holomorphic functions are infinitely many times differentiable, and a number of key results hold for them:

- Every derivative can be expressed in terms of the original function by means of Cauchy's formula: $f^{(n)}(a) = \frac{n!}{2\pi i} \int_{C(a,r)} \frac{f(z)}{(z-\zeta)^n} \,\mathrm{d}\zeta$.
- In particular, this yields bounds on the derivatives $f^{(n)}(a)$ in terms of the supremum norm of f.
- A holomorphic function is analytic, we have $f(z) = \sum_{n=1}^{\infty} \frac{f'(a)}{n!}(z-a)^n$ for z sufficiently close to a.
- In particular, we have unicity of power series: f is fully determined by its sequence of derivatives: if $f^{(n)}(a) = 0$ for all n, then f is zero in a neighborhood of a.
- Any holomorphic function on a suitably chosen domain in the complex plane can be approximated uniformly by rational functions.
- The maximum principle: a holomorphic function assumes its maximum on the boundary of its domain of definition.

A related class of functions that is an import object of study in potential theory is that of the *subharmonic functions*. These are upper semicontinuous functions that satisfy a mean value inequality. However, these functions are generally not continuous. To solve this problem, one might consider the *fine topology* on $\mathbb{C}$, the coarsest topology that makes all subharmonic functions continuous. As the name suggests, this topology is finer than the Euclidean.

In the 20th century, Bent Fuglede (see [1]) studied so-called *finely holomorphic functions*, which are obtained by taking derivatives with respect to this new topology. Clearly, ordinary holomorphic functions are finely holomorphic, but the converse does not hold. Nevertheless, the properties enumerated above all still hold in this setting. We will briefly discuss these results in chapter 1.

In the remainder of this thesis, we will consider a specific set (or class of sets) in the complex plane, and consider a topology on this set that is related to the fine topology. Here we continue the work by Gang Huang in [2], who introduced this example and a notion of holomorphy with respect to this topology. In chapter 2, we continue his investigation of this notion of holomorphy and show that most of the listed results still hold. In chapter 3, we see that there is a similar topology such that unicity of power series does *not* hold.

Chapter I

The fine topology

I.1 Basic definitions and results

Definition I.1.1. The *fine topology* on $\mathbb{C}$ is the coarsest topology that makes all subharmonic functions continuous.

As the name suggests, the following holds

Lemma I.1.2. *The fine topology is strictly finer than the Euclidean topology on* $\mathbb{C}$*.*

Proof. The function $z \mapsto \log|z-a|$ is subharmonic, hence $\{\log|z-a| < \log r\} = \{|z-a| < r\}$ is open in the fine topology for any a, r. Since there exist subharmonic functions that are not continuous, e.g. $v(z) = \sum_{n=1}^{\infty} 2^{-n} \log|z-2^{-n}|$ is subharmonic and discontinuous at 0, the converse does not hold. □

Now that we know that ordinary Euclidean balls are open in this topology, we might ask what sets are finely open that are not open in the Euclidean sense. Wiener's criterion is a useful tool in understanding fine open sets. It is based on the notion of logarithmic capacity.

Definition I.1.3. Let $E \subset \mathbb{C}$ be a set. We define

$$v(E) = \inf\{I(\mu) : \mu \in M_1^c(E)\},$$

where

$$I(\mu) = \iint \log \frac{1}{|z-w|} \,\mathrm{d}\mu(z)\,\mathrm{d}\mu(w)$$

is the *energy* of μ and $M_1^c(E)$ is the set of compactly supported probability measures on E. The *logarithmic capacity* of E is defined as $c(E) = e^{-v(E)}$.

The *outer logarithmic capacity* of $E \subset \mathbb{C}$ is defined by

$$c^*(E) = \inf\{c(F) : F \supset E \text{ open}\}.$$

This provides a notion of size of sets in the complex plane. While it is not a measure, it is increasing and subadditive. For more results and background on logarithmic capacity, see e.g. [6]. For certain sets, it can be computed explicitly.

Lemma I.1.4. *The logarithmic capacity of a ball $B(a, r)$ is r.*

Proof. See [6], Ch. 21, Corollary 10.3. □

Theorem I.1.5 (Wiener's criterion)**.** *Let $E \subset \mathbb{C}$ be a set, $a \in \mathbb{C}$ and $\lambda \in (0, 1/2]$. Set $E_n = E \cap \overline{A}(a, \lambda^{n+1}, \lambda^n)$. The point a is a fine limit point of E if and only if*

$$\sum_{n=1}^{\infty} \frac{n}{\log[c^*(E_n)^{-1}]} = \infty.$$

Proof. See [6], section 21.14. □

The following terminology is typically used:

Definition I.1.6. A set $E \subset \mathbb{C}$ is called *thick* at a point a if a is a fine limit point of a. If E is not thick at a, we say that E is *thin* at a.

Note that historically, a set is called E thin at a if there exists a subharmonic function v defined in a neighborhood of a such that $v(a) > \limsup_{z\to a} v(z)$.

In order to gain insight on the structure of fine open sets, we use this key theorem.

Theorem I.1.7. *A set $E \subset \mathbb{C}$ is thin at a if and only if $\mathbb{C} \setminus (E \setminus \{a\})$ is a fine neighborhood of a.*

Proof. See [7], Theorem I,3. □

We may now construct an example of a finely open set that is not open in the Euclidean topology.

Example I.1.8. Let $a_n = \frac{1}{2}\big(2^{-n} + 2^{-(n+1)}\big)$ and $r_n = 2^{-n^3}$. Set $U = B(0,1) \setminus \bigcup_{n=1}^{\infty} \overline{B(a_n, r_n)}$. Note that $0 \in U$ is not an (Euclidean) interior point. However, applying Wiener's criterion at $a = 0$ with $\lambda = 1/2$ to $E = \mathbb{C} \setminus U$, we have

$$\sum_{n=1}^{\infty} \frac{n}{\log\big[c^*(E_n)^{-1}\big]} = \sum_{n=1}^{\infty} \frac{n}{\log\big[r_n^{-1}\big]} = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty,$$

hence $\mathbb{C} \setminus U$ is thin at 0. By the above theorem, U is a fine neighborhood of zero. Clearly, by modifying the exponent in the radius, i.e. taking $r_n = 2^{-n^\delta}$, we see that the set obtained in this manner will be a fine neighborhood of zero if and only if $\delta \geq 2$.

I.2 Finely holomorphic functions

We now proceed to consider holomorphic functions with respect to the fine topology on $\mathbb{C}$. An obvious way to define these is by simply taking limits in the fine topology.

Definition I.2.1. A function $f\colon U \to \mathbb{C}$, where $U \subset \mathbb{C}$ is finely open, is said to be finely holomorphic if for every $z_0 \in U$ the fine limit

$$f'(z_0) := \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \tag{I.1}$$

exists and the function f' is finely continuous on U.

Note that the fine limit in (I.1) means that for every $\epsilon > 0$ there exists a fine neighborhood V of z_0 such that for all $z \in V \setminus \{z_0\}$ we have

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \epsilon.$$

The following theorem provides an alternative characterization of fine holomorphy.

Theorem I.2.2. *A function $f\colon U \to \mathbb{C}$ is finely holomorphic if and only if every $a \in U$ has a compact fine neighborhood $V \subset U$ such that $f|_V$ is the uniform limit of rational functions on V.*

It turns out that finely holomorphic functions behave much like ordinary holomorphic functions. Fuglede obtained the following results

Proposition I.2.3. *Let $f\colon U \to \mathbb{C}$ be finely holomorphic. Then the fine derivative f' is again a finely holomorphic function. As a consequence, finely holomorphic functions have infinitely many fine derivatives.*

Proof. See [1], Theorem 10. □

Theorem I.2.4. *A finely holomorphic function f is completely determined by its fine derivatives: if $f^{(n)}(z_0) = 0$ for all $n \geq 0$, then $f \equiv 0$ on a fine neighborhood of z_0.*

Proof. See [1], Theorem 14. □

The following rational approximation result holds for finely holomorphic functions.

Theorem I.2.5. *Let $f: U \to \mathbb{C}$ be finely holomorphic. Then every point in U has a compact fine neighborhood V such that there is a sequence g_j of rational functions on V with $\sup_{z\in V} |g_j^{(n)}(z) - f^{(n)}(z)| \to 0$ for all n.*

Proof. See [2], Theorem 3.2.2 or [1], Theorem 11. □

Since we will attempt to reproduce this proof in the next chapter for a slightly different setting, it is important to note that it is based on the following key lemma.

Lemma I.2.6. *Let $U \subset \mathbb{C}$ be a finely open and $K \subset \mathbb{C}$ compact. For any $\alpha \in \mathbb{R}$, the function*

$$h_\alpha(z) = \int_{K\setminus U} |z-\zeta|^{-\alpha}\, d\lambda(\zeta)$$

is finite and finely continuous everywhere in U.

Proof. See [1], Lemma 7. □

Chapter II

The illustrating example

II.1 Holomorphic functions on the illustrating example

Returning to the discussion at the end of Section I.1, we will now consider the case $\delta < 2$. As in [2], we consider the following set in the complex plane.

Definition II.1.1. Set $G = B(0,1) \setminus \bigcup_{n=1}^{\infty} \overline{B(a_n, r_n)}$, where $a_n = \frac{1}{2}\big(2^{-n} + 2^{-(n+1)}\big)$ and $r_n = 2^{-(n+1)^{15/8}}$. Note that $r_n \ll a_n$ and hence $0 \in G$. We have $G^\circ = G \setminus \{0\}$.

Let $\mathcal{B}$ be the collection of all Euclidean balls in G and all sets $B(0,R) \setminus \bigcup_{n=1}^{\infty} \overline{B(a_n, s_n)}$, where $R < 1$ and there exists $\delta \in (0, 7/8)$ such that $r_n \leq s_n \leq 2^{-(n+1)^{1+\delta}}$ for all n. It is easy to see that $\mathcal{B}$ is a basis for a topology. We call the topology generated by $\mathcal{B}$ the $\mathcal{G}$-topology.

As seen, the set G is neither an Euclidean nor a fine neighborhood of zero, c.f. Example I.1.8.

We will consider a notion of holomorphic functions on this set. There are a number of possible definitions for holomorphy; perhaps the most obvious one would be:

Definition II.1.2. A function $f \colon G \to \mathbb{C}$ is called $\mathcal{G}$-*holomorphic* if

$$\lim_{\zeta \to z} \frac{f(z) - f(\zeta)}{z - \zeta}$$

exists for all $z \in G$, where the limit is taken in the $\mathcal{G}$-topology. Clearly, for $z \neq 0$, this simply means that f is holomorphic (in the usual Euclidean sense) in z.

Alternatively, we might use this definition:

Definition II.1.3. We call a function f $\mathcal{G}$*-holomorphic* if every point in G has a $\mathcal{G}$-neighborhood U such that f is the uniform limit on $\bar{U}$ of a sequence of rational functions with poles off U.

We will show that these two definitions are equivalent. The following lemma is key to our argument.

Lemma II.1.4. *Let $U = B(0,R) \setminus \bigcup_{n=1}^{\infty} \overline{B(a_n, s_n)}$ be a $\mathcal{G}$-neighborhood of 0. There exists a $\mathcal{G}$-neighborhood $V \subset U$ of 0 such that the function*

$$h_\alpha(z) = \int_{\partial U} \frac{1}{|\zeta - z|^\alpha} \, d\zeta$$

is bounded on V for all $\alpha > 0$.

This lemma is the analogue of Lemma I.2.6 for the $\mathcal{G}$-set. Note however that here it suffices to consider integrals over the boundary of G, which simplifies the proof considerably.

Proof. We have $s_n \leq 2^{-(n+1)^{1+\delta}}$. Now set $V = B(0,T) \setminus \bigcup_{n=1}^{\infty} B(a_n, t_n)$, where $T < R$ and $t_n = 2^{-(n+1)^{1+\delta'}}$ for $\delta' \in (0, \delta)$. For $z \in V$ we have

$$\begin{aligned} h_\alpha(z) &= \int_{C(0,R)} \frac{1}{|\zeta - z|^\alpha} \, \mathrm{d}\zeta + \sum_{n=1}^{\infty} \int_{C(a_n, s_n)} \frac{1}{|\zeta - z|^\alpha} \, \mathrm{d}\zeta \\ &\leq C_\alpha + \sum_{n=1}^{\infty} \frac{2\pi s_n}{(t_n - s_n)^\alpha} \end{aligned}$$

Hence it suffices to show that

$$\sum_{n=1}^{\infty} \frac{s_n}{(t_n - s_n)^\alpha} = \sum_{n=1}^{\infty} \frac{s_n / t_n^\alpha}{(1 - s_n/t_n)^\alpha} \leq C_\alpha \sum_{n=1}^{\infty} \frac{s_n}{t_n^\alpha} < \infty,$$

where we use the fact that $\sup_n s_n/t_n < 1$. For this choice of s_n and t_n, the latter sum is bounded by

$$\sum_{n=1}^{\infty} 2^{-n^{1+\delta} + \alpha n^{1+\delta'}} \leq \sum_{n=1}^{\infty} 2^{-Cn^{1+\delta}} < \infty$$

if $\delta > \delta'$. The result follows. □

We can now regain some standard results about holomorphic functions, using either definition.

Proposition II.1.5 (Cauchy's formula)**.** *Let f be a $\mathcal{G}$-holomorphic function in the sense of Definition II.1.2. Then there exists a $\mathcal{G}$-neighborhood of* 0 *such that for all $z \in U$ we have*

$$f(z) = \frac{1}{2\pi i}\int_{\partial U} \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

Remark II.1.6. The use of ∂U here is perhaps ambiguous, since we can take the boundary in either the Euclidean topology or in the $\mathcal{G}$-topology. However, since these differ by only one point, it does not change the integral.

Proof. Note that f is holomorphic away from zero and hence the result reduces to Cauchy's formula for holomorphic functions for $z \neq 0$. Let U be a neighborhood of zero such that f is bounded on $\overline{U}$. Then for $z \in U$ we have

$$\begin{aligned}\left|\int_{\partial U} \frac{f(\zeta)}{\zeta}\,\mathrm{d}\zeta - \int_{\partial U} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta\right| &\leq |z| \int_{\partial U} \frac{|f(\zeta)|}{|\zeta||\zeta - z|}\,\mathrm{d}\zeta \\ &\leq |z|\|f\|_{\infty,U}\left(\int_{\partial U}\frac{1}{|\zeta|^2}\,\mathrm{d}\zeta \int_{\partial U}\frac{1}{|\zeta - z|^2}\,\mathrm{d}\zeta\right)^{1/2},\end{aligned}$$

where in the last step we apply Cauchy-Schwartz. By Lemma II.1.4, the integrals on the right are bounded. Letting $z \to 0$ in U, we see that

$$\int_{\partial U} \frac{f(\zeta)}{\zeta}\,\mathrm{d}\zeta = \lim_{z\to 0}\int_{\partial U}\frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta = \lim_{z \to 0} 2\pi i f(z) = 2\pi i f(0). \qquad \square$$

We may now see that Definition II.1.2 implies II.1.3 (c.f. Theorem 10.5 in [8]).

Theorem II.1.7. *Let f be a $\mathcal{G}$-holomorphic function in the sense of Definition II.1.2. Then every point in $z \in \overline{G}$ has a $\mathcal{G}$-neighborhood V such that f is the uniform limit on $\overline{V}$ of a sequence of rational functions with poles off V.*

Proof. Let U be a neighborhood on which Lemma II.1.4 and Cauchy's formula hold and let $V = B(0,R) \setminus \bigcup_{n=1}^{\infty} B(a_n, t_n) \subset U$ be a smaller neighborhood. Then for $z \in K = \overline{V}$, we have

$$f(z) = \frac{1}{2\pi i}\int_{\partial U}\frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta.$$

To proceed, we adapt the proof of Runge's theorem (c.f. Theorem 13.6 in [5]). Let R be the set of rational functions with poles off K. We want to show that $f \in \overline{R} \subset C(K)$. By Hahn-Banach and the fact that the dual of

$C(K)$ is the space $M(K)$ of all complex regular Borel measures on K, it is sufficient to show that for each $\mu \in M(K)$ such that $\int g \, \mathrm{d}\mu = 0$ for all $g \in R$, we have $\int f \, \mathrm{d}\mu = 0$. Applying Fubini's theorem, we have

$$2\pi i \int_K f(z) \, \mathrm{d}\mu(z) = \int_K \left(\int_{\partial U} \frac{f(\zeta)}{\zeta - z} \, \mathrm{d}\zeta \right) \mathrm{d}\mu(z)$$
$$= \int_{\partial U} f(\zeta) \left(\int_K \frac{1}{\zeta - z} \, \mathrm{d}\mu(z) \right) \mathrm{d}\zeta.$$

Note that the use of Fubini here is justified since

$$\int_K \int_\gamma \frac{|f(\zeta)|}{|\zeta - z|} \, \mathrm{d}\zeta \, \mathrm{d}\mu(z) \le \|f\|_{\infty, \overline{U}} \int_K \int_{\partial U} \frac{1}{|\zeta - z|} \, \mathrm{d}\zeta \, \mathrm{d}\mu(z)$$

is finite by Lemma II.1.4. By assumption, the function $\hat{\mu}(\zeta) = \int_K \frac{1}{\zeta - z} \, \mathrm{d}\mu(z)$ is identically zero on $\mathbb{C} \setminus K$ and in particular on ∂U. Hence $\int f \, \mathrm{d}\mu = 0$ as desired. □

For the converse, we will again require Cauchy's formula.

Proposition II.1.8 (Cauchy's formula)**.** *Let f be a $\mathcal{G}$-holomorphic function in the sense of Definition II.1.3. Then for all $z \in G$ we have*

$$f(z) = \frac{1}{2\pi i} \int_{\partial G} \frac{f(\zeta)}{\zeta - z} \, d\zeta.$$

Proof. Note that f is again holomorphic away from zero and hence the result holds for $z \neq 0$. Let f_n be a sequence of rational functions such that $f_n \to f$ uniformly on a neighborhood $U = B(0, R) \setminus \bigcup_{n=1}^\infty \overline{B(a_n, s_n)}$ of 0. By Cauchy's theorem, we have

$$\int_{\partial G} \frac{f(\zeta)}{\zeta} \, \mathrm{d}\zeta = \left(\int_{C(0,1)} + \sum_{n=1}^\infty \int_{C(a_n, r_n)} \right) \frac{f(\zeta)}{\zeta} \, \mathrm{d}\zeta$$
$$= \left(\int_{C(0,R)} + \sum_{n=1}^\infty \int_{C(a_n, s_n)} \right) \frac{f(\zeta)}{\zeta} \, \mathrm{d}\zeta = \int_{\partial U} \frac{f(\zeta)}{\zeta} \, \mathrm{d}\zeta.$$

Note that

$$\left| \int_{\partial U} \frac{f_n(\zeta)}{\zeta} \, \mathrm{d}\zeta - \int_{\partial U} \frac{f(\zeta)}{\zeta} \, \mathrm{d}\zeta \right| \le \|f_n - f\|_{\infty, U} \int_{\partial U} \frac{1}{|\zeta|} \, \mathrm{d}\zeta$$

and the last integral is finite by Lemma II.1.4. Hence

$$f(0) = \lim_{n \to \infty} f_n(0) = \lim_{n \to \infty} \int_{\partial G} \frac{f_n(\zeta)}{\zeta} \, \mathrm{d}\zeta = \int_{\partial G} \frac{f(\zeta)}{\zeta} \, \mathrm{d}\zeta.$$ □

Theorem II.1.9. *Let f be a $\mathcal{G}$-holomorphic function in the sense of Definition II.1.3 and let (f_j) be a sequence of rational functions such that $f_j \to f$ uniformly on the closure of $U = B(0,R) \setminus \overline{\bigcup_{n=1}^\infty B(a_n, s_n)}$. Then there is a $\mathcal{G}$-neighborhood $V \subset U$ of 0 such that $f_j^{(n)} \to f^{(n)}$ uniformly on V for each $n \in \mathbb{Z}_{\geq 0}$, where*

$$f^{(n)}(z) := \frac{n!}{2\pi i} \int_{\partial G} \frac{f(\zeta)}{(z-\zeta)^{n+1}}\, d\zeta.$$

Furthermore, f is infinitely often $\mathcal{G}$-differentiable with derivatives $f^{(n)}$.

Proof. The first claim follows directly from Lemma II.1.4 by applying Cauchy's formula to the f_j. Indeed, using Cauchy's theorem we see that for $z \in V$

$$\begin{aligned}
2\pi\left|f_j^{(n)}(z) - f^{(n)}(z)\right| &= n! \left| \int_{\partial G} \frac{f_j(z)}{(\zeta - z)^{n+1}}\, \mathrm{d}\zeta - \int_{\partial G} \frac{f(\zeta)}{(z-\zeta)^{n+1}}\, \mathrm{d}\zeta \right| \\
&= n! \|f_j - f\|_{\infty,U} \int_{\partial U} \frac{1}{|\zeta - z|^{n+1}}\, \mathrm{d}\zeta \\
&\leq C_n n! \|f_j - f\|_{\infty,U} \to 0.
\end{aligned}$$

Since f is holomorphic away from zero, the differentiability is clear for $z \neq 0$. By Cauchy's formula, we have

$$\begin{aligned}
2\pi \left| \frac{f(z) - f(0)}{z} - f^{(1)}(0) \right| &= \left| \int_{\partial G} \left[\frac{f(\zeta)}{z(\zeta - z)} - \frac{f(\zeta)}{z\zeta} - \frac{f(\zeta)}{\zeta^2} \right] \mathrm{d}\zeta \right| \\
&= \left| \int_{\partial U} \frac{f(\zeta) z^2}{z(\zeta - z)\zeta^2}\, \mathrm{d}\zeta \right| \\
&\leq \|f\|_{\infty,U} |z| \int_{\partial U} \frac{1}{|\zeta - z||\zeta|^2}\, \mathrm{d}\zeta \\
&\leq \|f\|_{\infty,U} |z| \left[\int_{\partial U} \frac{1}{|\zeta - z|^2}\, \mathrm{d}\zeta \int_{\partial U} \frac{1}{|\zeta|^4}\, \mathrm{d}\zeta \right]^{1/2}.
\end{aligned}$$

By another application of Lemma II.1.4, these integrals are bounded for $z \in V$. The result follows. □

II.2 The maximum principle

We now consider the maximum principle for $\mathcal{G}$-holomorphic functions. It is clear that any such function cannot have a local maximum at $0 \neq z \in G$, hence it remains to show that 0 cannot be a local maximum. We will need the following lemma.

Lemma II.2.1. *Let f be a $\mathcal{G}$-holomorphic function such that $f(G) \subset K$, where $K \subset \mathbb{C}$ is compact such that $\mathbb{C}\backslash K$ is connected. Suppose (g, K) satisfies the assumptions of Mergelyan's Theorem A.1. Then $g \circ f$ is $\mathcal{G}$-holomorphic.*

Proof. Let (g_n) be a sequence of polynomials such that $g_n \to g$ uniformly on K and (f_n) a sequence of rational functions such that $f_n \to f$ uniformly on U. We have

$$\left|g_n\big(f_n(z)\big) - g\big(f(z)\big)\right| \leq \left|g_n\big(f_n(z)\big) - g\big(f_n(z)\big)\right| + \left|g\big(f_n(z)\big) - g\big(f(z)\big)\right|.$$

The first term tends to zero uniformly in z by the uniform convergence of g_n; the second term by uniform convergence of f_n and uniform continuity of g on K. □

Theorem II.2.2. *Let f be a $\mathcal{G}$-holomorphic function. Then f does not have a local maximum at 0.*

Proof. Suppose f does have a local maximum in 0. Without loss of generality, we may assume $f(0) = 1$. Then $|f(z)| \leq 1$ on some $\mathcal{G}$-open U and f maps U into the unit ball. By the Riemann mapping Theorem A.2, we can find a holomorphic function g such that $|h(z)| := |(g \circ f)(z)| < 1$ for all $z \in U$ such that $f(z) \neq 1$ and $h(0) = 1$. By Lemma II.2.1, h is $\mathcal{G}$-holomorphic. Then $z \mapsto h(z)^n$ is also $\mathcal{G}$-holomorphic and hence

$$1 = \lim_{n\to\infty} h(0)^n = \lim_{n\to\infty} \frac{1}{2\pi i} \int_{\partial G} \frac{h(\zeta)^n}{\zeta}\, \mathrm{d}\zeta = \frac{1}{2\pi} \int_{\partial G} \frac{1_{f(\zeta)=1}}{\zeta}\, \mathrm{d}\zeta,$$

where we apply the dominated convergence theorem to justify interchanging limit and integral. Since f is holomorphic on the annulus $A(a_n, r_n, r_n + \epsilon_n)$ for each n, the set of points on the circle $C(a_n, r_n)$ where it is equal to one has measure zero by Corollary A.4. We conclude that the integral on the right hand side is zero, which is a contradiction. □

II.3 Size of the coefficients

We now consider the power series at 0 of a $\mathcal{G}$ and derive bounds on the coefficients $f^{(n)}(0)$.

Lemma II.3.1. *Let f be a $\mathcal{G}$-holomorphic function. Then the coefficients $c_\alpha = f^{(\alpha)}(0)$ satisfy $c_\alpha = O\left(\alpha! \sum_{n=1}^{\infty} \frac{r_n}{a_n^\alpha}\right)$. For our choice of r_n, we obtain $c_\alpha = O(\alpha! \alpha^{8/7} 2^{\alpha^{15/7}})$.*

Proof. We have

$$\frac{|f^{(\alpha)}(0)|}{\alpha!} \leq \frac{1}{2\pi}\int_{\partial G}\frac{|f(\zeta)|}{|\zeta - z|^\alpha}\,\mathrm{d}\zeta \leq \frac{\|f\|_\infty}{2\pi}\sum_{n=1}^{\infty}\int_{C(a_n,r_n)}\frac{1}{|\zeta|^\alpha}\,\mathrm{d}\zeta$$
$$\leq \|f\|_\infty \sum_{n=1}^{\infty}\frac{r_n}{(a_n - r_n)^\alpha} \leq \|f\|_\infty C\sum_{n=1}^{\infty}\frac{r_n}{a_n^\alpha}.$$

We consider

$$\sum_{n=1}^{\infty}\frac{2^{-n^{15/8}}}{2^{-\alpha n}} = \sum_{n=1}^{\infty}2^{\alpha n - n^{15/8}}.$$

Note that $\alpha n - n^{15/8}$ is maximal at $n = c\alpha^{8/7}$ (where $c = \left(\frac{8}{15}\right)^{8/7}$) with maximal value $C\alpha^{15/7}$. Furthermore, we have $\alpha n - n^{15/8} \leq -n$ for $n > (1+\alpha)^{8/7}$. We conclude that

$$\sum_{n=1}^{\infty}2^{\alpha n - n^{15/8}} \leq 1 + c(1+\alpha)^{8/7}2^{\alpha^{15/7}} = O\big(\alpha^{8/7}2^{\alpha^{15/7}}\big). \qquad \square$$

On the other hand, let $f(z) = \sum_{n=1}^{\infty}\frac{b_n}{a_n - z}$, then

$$f^{(\alpha)}(z) = \alpha!\sum_{n=1}^{\infty}\frac{b_n}{(a_n - z)^\alpha}.$$

If we set $b_n = \eta_n r_n$, where η_n is some sequence with $\eta_n \to 0$, this is a $\mathcal{G}$-holomorphic function by the below lemma. Furthermore, we have

$$f^{(\alpha)}(0) = \alpha!\sum_{n=1}^{\infty}\frac{b_n}{a_n^\alpha} = \alpha!\sum_{n=1}^{\infty}\eta_n\frac{r_n}{a_n^\alpha}.$$

Since we may choose η_n to tend to zero arbitrarily slowly, we see that the above bound is sharp.

Lemma II.3.2. *Let*

$$f_N(z) = \sum_{n=1}^{N}\frac{b_n}{a_n - z}.$$

Then $f_N(z)$ converges uniformly on a $\mathcal{G}$-neighborhood of 0 (and hence $f = \lim f_N$ is a $\mathcal{G}$-holomorphic function) if and only if there exists a sequence (s_n) with $r_n \leq s_n \leq 2^{-(n+1)^{1+\delta}}$ such that $\frac{|b_n|}{s_n} \to 0$.

Proof. We will first show that if $\frac{|b_n|}{s_n} \to 0$, then the series converges uniformly on the closure of $U = B(0,R) \setminus \bigcup_{n=1}^{\infty} \overline{B(a_n, s_n)}$.

Let $\epsilon > 0$. Suppose $z \in \bar{U}$ has real part between a_m and a_{m+1} with $m > N$ for some number N which we will choose later on. We have

$$\sum_{n \geq N} \frac{|b_n|}{|a_n - z|} \leq \frac{|b_m|}{s_m} + \frac{|b_{m+1}|}{s_{m+1}} + \sum_{n=N}^{m-1} \frac{|b_n|}{a_n - a_m} + \sum_{n \geq m+2} \frac{|b_n|}{a_m - a_n}.$$

If $a_n = c2^{-n}$, then

$$\sum_{n=N}^{m-1} \frac{|b_n|}{c2^{-n} - c2^{-m}} \leq \frac{1}{c} \sum_{n=N}^{m-1} \frac{|b_n|}{2^{-(n+1)}} = \frac{1}{c} \sum_{n=N}^{m-1} \frac{s_n}{2^{-(n+1)}} \frac{|b_n|}{s_n}.$$

Note that $\sum_n \frac{s_n}{2^{-(n+1)}}$ is a convergent series and $|b_n|/s_n \to 0$, hence we may choose N such that this sum is less than ϵ. We consider the second sum:

$$\frac{1}{c} \sum_{n \geq m+2} \frac{|b_n|}{2^{-m} - 2^{-n}} \leq \frac{1}{c} \sum_{n \geq m+2} \frac{|b_n|}{2^{-(m+1)}} = \frac{2^{m+1}}{c} \sum_{n \geq m+2} n^2 s_n \frac{1}{n^2} \frac{|b_n|}{s_n}.$$

Note that $2^{n+1} n^2 s_n \to 0$, hence we may choose N such that $n^2 s_n < \epsilon 2^{-(n+1)} < \epsilon 2^{-(m+1)}$. Since $\sum_{n \geq m+2} \frac{1}{n^2} \frac{|b_n|}{s_n}$ is convergent, we may choose N such that this is less than ϵ. Clearly, N can be chosen such that $\frac{|b_m|}{s_m} + \frac{|b_{m+1}|}{s_{m+1}} < \epsilon$ for all $m > N$. We conclude that $\sum_{n \geq N} \frac{|b_n|}{|a_n - z|} < 4\epsilon$ for all $z \in G$ with real part between 0 and a_N. If $\Re z > a_N$ or $\Re z < 0$, all terms in $\sum_{n \geq N} \frac{|b_n|}{|a_n - z|}$ are even smaller and the result also holds. Hence the series converges uniformly on $\overline{U}$.

Now suppose $\sum_{n=1}^{\infty} \frac{b_n}{a_n - z}$ converges unformly on $\bar{U}$. Taking $z = a_m + s_m$, we have

$$\left| \sum_{n \geq m} \frac{b_n}{a_n - z} \right| \geq \frac{|b_m|}{s_m} - \left| \sum_{n \geq m+1} \frac{b_n}{a_m - a_n} \right|.$$

In the above, we have seen that the second term tends to zero as $m \to \infty$. The left hand side also tends to zero, hence we must have $|b_m|/s_m \to 0$. $\square$

Chapter III

More general radius

III.1 $\mathcal{G}$-sets

In the previous chapter, we used a fairly specific choice of radius and center for the balls $B(a_n, r_n)$. There is no immediate reason for this choice. Indeed, looking at the proof of Lemma II.1.4, we see that we only need fairly mild conditions on the radii r_n and s_n. The proof of Theorem II.1.9 does not depend on the choice of radius. For example, we might consider the following variant.

Definition III.1.1. Set $G_\ell = B(0,1) \setminus \bigcup_{n=2}^{\infty} \overline{B(a_n, r_n)}$, where $a_n = 2^{-n}$ and $r_n = 2^{-\frac{1}{4}\sqrt{n}\log_2 n}$. Note that $r_n \ll a_n$ and hence $0 \in G_\ell$. We have $G_\ell^\circ = G_\ell \setminus \{0\}$.

Let $\mathcal{B}$ be the collection of all Euclidean balls in G_ℓ and all sets $B(0,R) \setminus \bigcup_{n=1}^{\infty} \overline{B(a_n, s_n)}$, where $R < 1$ and $r_n \leq s_n \leq 2^{-\frac{1}{4}n^\delta \log n}$ for some $\delta \in (0,1)$. It is easy to see that $\mathcal{B}$ is a basis for a topology. We call the topology generated by $\mathcal{B}$ the $\mathcal{G}_\ell$-topology.

By a straightforward calculation, we see that the key lemma still holds in this situation.

Lemma III.1.2. *Let $U = B(0,R) \setminus \bigcup_{n=1}^{\infty} \overline{B(a_n, s_n)}$ be a $\mathcal{G}$-neighborhood of 0. There exists a $\mathcal{G}$-neighborhood $V \subset U$ of 0 such that the function*

$$h_\alpha(z) = \int_{\partial G_\ell} \frac{1}{|\zeta - z|^\alpha} \, d\zeta$$

is bounded on V for all $\alpha > 0$.

Proof. We follow the proof of Lemma II.1.4, where $s_n \leq 2^{-\frac{1}{4}n^\delta \log_2 n}$ and $t_n = 2^{-\frac{1}{4}n^{\delta'} \log_2 n}$ for $\delta' \in (0, \delta)$. Then

$$\sum_{n=1}^{\infty} \frac{s_n}{t_n^\alpha} \leq \sum_{n=1}^{\infty} 2^{\frac{1}{4}(-n^\delta + \alpha n^{\delta'}) \log_2 n} \leq \sum_{n=1}^{\infty} 2^{-cn^\delta \log_2 n} < \infty. \qquad \square$$

III.2 Unicity of power series

We now turn to the question of unicity of power series. Any $\mathcal{G}$-holomorphic function has infinitely many $\mathcal{G}$-derivatives at 0. We might ask if Theorem I.2.4 still holds for these functions and how this depends on the choice of r_n. For $r_n = 2^{-\frac{1}{4}n \log_2 n}$, the answer is no. Note that it can be seen that the proof of II.2.2 is still valid for this choice of r_n, and hence the maximum principle *does* hold.

Theorem III.2.1. *There exists a $\mathcal{G}_\ell$-holomorphic function f such that $f^{(n)}(0) = 0$ for all n, but $f \neq 0$.*

Proof. Let $f_0(z) = \frac{1}{1-z} - \frac{1}{1-2z}$. Note that $f_0(0) = 0$. Define $f_1(z) = f_0(z) - \frac{1}{4} f_0(4z)$, so that $f_1(0) = f_1'(0) = 0$. Then $f_2(z) = f_1(z) - \frac{1}{16^2} f_1(16z)$ has the first two derivatives equal to zero and we proceed inductively with

$$f_n(z) = f_{n-1}(z) - \frac{1}{2^{n2^n}} f_{n-1}\big(2^{2^n} z\big).$$

Note that this function has $f^{(k)}(0) = 0$ for all $k \leq n$. Furthermore, we have $f_{n-1}(z) = \sum_{k=0}^{2^n-1} \alpha_k \frac{2^{-k}}{2^{-k}-z}$ for some coefficients (α_k). These coefficients satisfy

$$\left|\alpha_{2^k-1}\right| = \prod_{j=0}^{k-1} \frac{1}{2^{j2^j}} = 2^{-\sum_{j=0}^{k-1} j2^j} \leq 2^{-(k-1)2^{k-1}}$$

and $(|\alpha_\ell|)$ is a nonincreasing sequence. For each ℓ, let $k = \lfloor \log_2(\ell+1) \rfloor \leq \ell$. Then

$$|\alpha_\ell| \leq \left|\alpha_{2^k-1}\right| \leq 2^{-(k-1)2^{k-1}} \leq 2^{-(\log_2(\ell+1)-2)(\ell+1)/4},$$

where we use $k \geq \log_2(\ell+1) - 1$. We now apply Lemma II.3.2 with $b_n = \alpha_n 2^{-n}$ and $r_n = 2^{-\frac{1}{4}n \log n}$, we have

$$\begin{aligned}\log_2 \frac{b_n}{r_n} &\leq n \log_2 n - n - (\log_2(n+1) - 2)\,(n+1)/4 \\ &= \tfrac{1}{4} n \log_2 n - \tfrac{1}{4}(n+1)\log_2(n+1) - \tfrac{1}{2}n + \tfrac{1}{4} \to -\infty. \qquad \square\end{aligned}$$

Note that this results requires sufficiently large values of r_n. For example, for our original choice of $r_n = 2^{-(n+1)^{15/8}}$, the argument fails. The question remains whether or not unicity holds for smaller values of r_n.

We turn to quasi-analytic classes in an attempt to resolve this question, see Definition A.5. If we can prove that $\mathcal{G}$-holomorphic functions form a quasi-analytic class, we are done. Sadly, this turns out not to be the case (cf. [3]).

Proposition III.2.2. *There exists a $\mathcal{G}$-holomorphic function such that its restriction to the imaginary axis is not in any quasi-analytic class.*

Proof. Let

$$f(z) = \sum_{n=1}^{\infty} \frac{1}{n^2} \frac{r_n}{a_n - z}.$$

By Lemma II.3.2, f is $\mathcal{G}$-holomorphic. Note that

$$f^{(k)}(0) = k! \sum_{n=1}^{\infty} \frac{r_n}{n^2 a_n^k} = k! \sum_{n=1}^{\infty} \frac{1}{n^2} \frac{2^{kn}}{2^{(n+1)^{15/8}}}$$

For sufficiently large k, we have

$$\begin{aligned} \left|f^{(k)}(0)\right| &\geq k! \sum_{n=1}^{k-1} \frac{1}{n^2} \frac{2^{kn}}{2^{(n+1)^{15/8}}} \geq \frac{k!}{k^2 2^{k^{15/8}}} \sum_{n=1}^{k-1} 2^{kn} = \frac{k!}{k^2 2^{k^{15/8}}} \frac{2^{k^2} - 2^k}{2^k - 1} \\ &= \frac{k!}{k^2 2^{k^{15/8}}} \frac{2^{k^2-k} - 1}{1 - 2^{-k}} \geq \frac{k!}{k^2 2^{k^{15/8}}} 2^{k^2-k-1} = \frac{k!}{k^2} 2^{k^2-k^{15/8}-k-1}. \end{aligned}$$

Then

$$\left(\frac{k!}{k^2} 2^{k^2-k^{15/8}-k-1}\right)^{1/k} \geq \left(2^{k^2-k^{15/8}-k-1}\right)^{1/k} = 2^{k-k^{7/8}-1-\frac{1}{k}} \geq 2^{Ck}$$

for some C and k large. We have $\sum_{k=1}^{\infty} \frac{1}{2^{Ck}} < \infty$, hence by the Denjoy-Carleman theorem A.6, there is no quasi-analytic class that contains f.

□

Appendix A

Some results from (complex) analysis

In this appendix we state a number of well-known results from complex analysis which are used throughout the thesis.

Theorem A.1 (Mergelyan)**.** *Let $K \subset \mathbb{C}$ be compact with $\mathbb{C} \setminus K$ connected. If $f\colon K \to \mathbb{C}$ is continuous on K and holomorphic on K°, then f can be approximated uniformly on K by polynomials.*

Proof. See [5], Theorem 20.5. □

Theorem A.2 (Riemann mapping theorem)**.** *Let U be a simply connected domain in $\mathbb{C}$ with $U \neq \mathbb{C}$. Then for every $a \in U$ there is precisely one bijective holomorphic $g\colon U \to B(0,1)$ with holomorphic inverse such that $g(a) = 0$ and $g'(a) > 0$.*

Proof. See [5], Theorem 14.8. □

Theorem A.3. *Let f be continuous on $\overline{B(0,1)}$ and holomorphic on the interior. Then $\{f = 0\} \cap C(0,1)$ has one-dimensional Lebesgue measure zero in $C(0,1)$.*

Proof. See [5], Theorem 17.18. □

Corollary A.4. *Let $U \subset \mathbb{C}$ be a simply connected domain with f continuous on $\overline{U}$ and holomorphic on U. Suppose ∂U is a Jordan curve. Then $\{f = 0\} \cap \partial U$ has one-dimensional Lebesgue-measure zero in ∂U.*

Definition A.5. *Let $M = (M_n)$ be a sequence of positive real numbers. Define $C(M)$ to be the set of $C^\infty(\mathbb{R})$ functions f such that there exist constants β_f and B_f satisfying*

$$\|f^{(n)}\|_\infty \leq \beta_f B_f^n M_n$$

for all $n \in \mathbb{Z}_{\geq 0}$*. We say that* $C(M)$ *is* quasi-analytic *if it does not contain any function* $f \neq 0$ *such that* $f^{(n)}(0) = 0$ *for all* n*.*

Theorem A.6 (Denjoy-Carleman)**.** *Let* $M = (M_n)$ *be an increasing sequence of positive real numbers with* $M_0 = 1$*. The set* $C(M)$ *is a quasi-analytic class if and only if* $\sum_{n=0}^{\infty} \frac{1}{M_n^{1/n}} = \infty$*.*

Proof. See [5], Theorem 19.11 (p. 380). □

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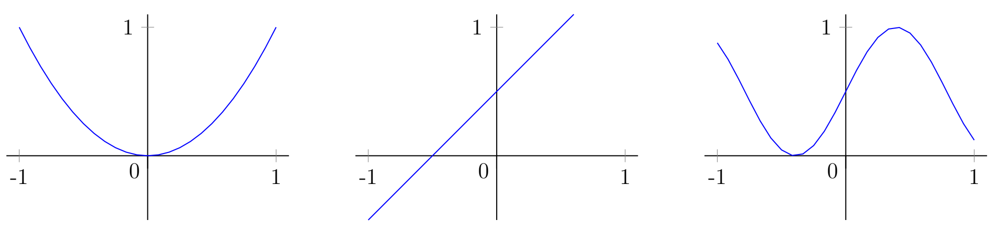
Populaire samenvatting

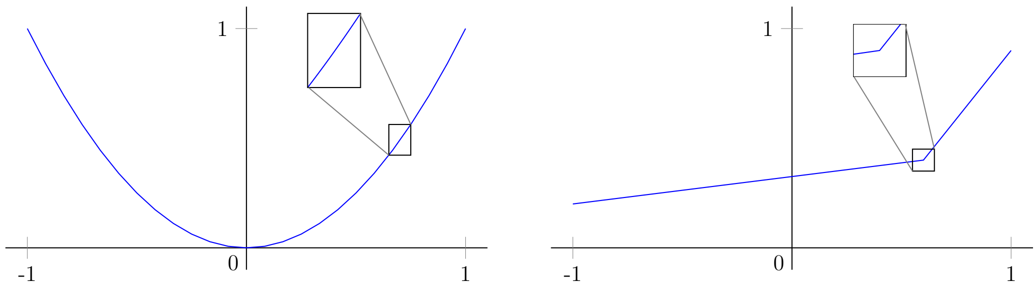
In de analyse houdt men zich bezig met het bestuderen van functies; wiskundige objecten die aan één getal een ander getal toevoegen. In het meest eenvoudige geval, gaat dit om reële getallen en kan men deze functies weergeven in een grafiek, zie Figuur 1. *Lineaire functies* zijn functies waarvan de grafiek een lijn is; zij worden gegeven door een functievoorschrift als $f(x) = 3x - 2$ (hier is 3 de *helling* van de lijn en -2 een *verschuiving*). Deze functies zijn bijzonder makkelijk om mee te werken, maar helaas zijn de meeste functies niet lineair.

Om toch gebruik te kunnen maken van de gunstige eigenschappen van lineaire functies, bekijkt men functies die *differentieerbaar* zijn: functies die lokaal op lineaire functies lijken, d.w.z. bij voldoende inzoomen lijkt de grafiek op een rechte lijn, zie Figuur 2. Wiskundig preciezer betekent dit dat voor ieder punt op de horizontale as er een kleine omgeving om dat punt heen is, waarin de functie goed benaderd wordt door een lineaire functie.

De functietheorie is een vakgebied binnen de analyse waar men niet functies op de reële lijn bestudeert, maar functies van en naar het complexe vlak $\mathbb{C}$. Dit is een twee-dimensionaal vlak, waar er niet alleen een natuurlijke optelling is gedefinieerd, maar waar het ook mogelijk is om twee punten met elkaar te vermenigvuldigen (zie Figuur 3). Op het complexe vlak kunnen we ook functies als $f(z) = 2z + 1$ of $g(z) = z^2$ bekijken. Het symbool z stelt hier een punt in het vlak voor, d.w.z. z heeft een x en een y-coördinaat.



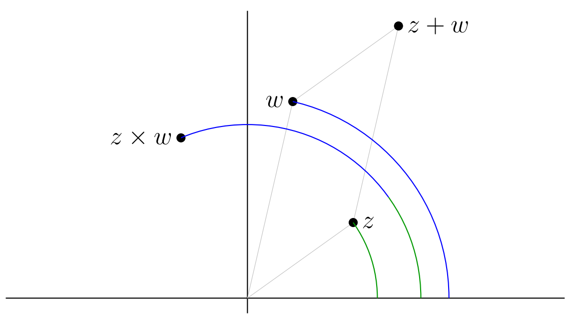
Figuur 1: Grafieken van $f(x) = x^2$, $g(x) = x + \frac{1}{2}$ en $h(x) = \sin(4x)/2 + \frac{1}{2}$



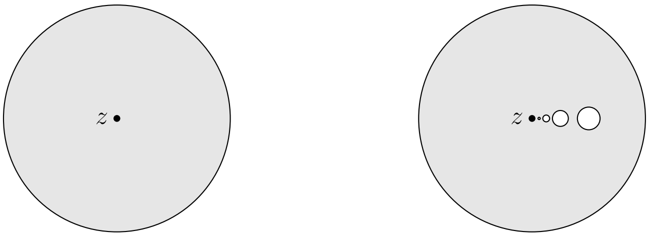
Figuur 2: Grafiek van een functie die overal differentieerbaar is (links) en een functie die in één punt niet differentieerbaar is (rechts)

Ook in het complexe vlak kunnen we kijken naar functies die differentieerbaar zijn, oftewel lokaal goed te benaderen door een (complexe) lineaire functie als $f(z) = 2z + 1$. Men spreekt van *holomorfe* functies, en deze functies blijken een aantal verassende eigenschappen te hebben, die differentieerbare functies in het reële geval niet hebben. Een dergelijke functie is dus in een omgeving van ieder punt in het complexe vlak goed te benaderen door een lineaire functie. De meest voor de hand liggende omgevingen om te bekijken zijn kleine bolletjes om punten heen, zie Figuur 4. Deze omgevingen beschrijven een *topologie*, de zogenaamde Euclidische topologie. Zij geven betekenis aan het begrip "dichtbij".

In de jaren tachtig van de vorige eeuw is in het kader van de potentiaaltheorie de *fijne topologie* ontwikkeld. In deze topologie zijn er veel vreemdere omgevingen, zoals te zien in Figuur 4. Deze omgeving is ook een klein bolletje



Figuur 3: Optelling en vermenigvuldiging in het complexe vlak



Figuur 4: Een omgeving van een punt z in de Euclidische (links) en de fijne topologie (rechts)

om het punt, maar dan met een (oneindig lange) rij nog veel kleinere bollen eruit weggelaten. Als men volgens deze topologie gaat kijken naar differentieerbaarheid, zijn er meer functies holomorf. Immers moet de functie nu op een kleiner stuk goed te benaderen zijn door een lineaire functie. Hoe groter de bollen zijn die we weglaten, hoe meer holomorfe functies er mogelijk zijn.

Voor de fijne topologie moeten de weggelaten bollen zeer klein zijn, met stralen $1/2^2, 1/2^9, 1/2^{16}, \ldots$ naarmate de bollen dichter bij z komen. In deze scriptie worden een aantal topologieën bestudeerd, waarbij deze stralen groter genomen worden. Er wordt onderzocht wat voor holomorfe functies er op deze manier verkregen worden, en of de eerder genoemde eigenschappen uit het Euclidische geval hier nog steeds gelden.