WISKUNDIGE LOGICA (2016-2017)
HOMEWORK 1

- Deadline: February 28 — at the **beginning** of class.
- Homework can be submitted electronically (in a single pdf-file!) to Jolien Oomens.
- Grading is from 0 to 100 points.
- Success!

(1) (10pt) Compute

- free($\exists x_1 Rx_1x_2x_3 \wedge \forall y Py$),
- free($\exists x_1 Qx_1x_2 \rightarrow Qx_1x_4$),
- SF($\exists x_1 Rx_1x_2x_3 \wedge \forall y Sy$),
- var($(x + f(x,y)) \cdot g(z,y)$).

(2) (20pt) Define inductively the function Var which associates with each formula the set of variables occurring in it. (Note that var for terms is defined in Defition 4.5 of the book.)

(3) (20pt) Let $S = \{R\}$. Let also $\mathbf{N}$, $\mathbf{Z}$, and $\mathbf{Q}$ be the sets of natural numbers, integers and rationals, respectively.

(a) Write an S-sentence which is satisfied in $(\mathbf{N}, <^{\mathbf{N}})$ and not satisfied in $(\mathbf{Z}, <^{\mathbf{Z}})$. Vice versa write an S-sentence which is satisfied in $(\mathbf{Z}, <^{\mathbf{Z}})$ and not satisfied in $(\mathbf{N}, <^{\mathbf{N}})$. Justify your solutions.

(b) Write an S-sentence which is satisfied in $(\mathbf{Z}, <^{\mathbf{Z}})$ and not satisfied in $(\mathbf{Q}, <^{\mathbf{Q}})$. Vice versa write an S-sentence φ which is satisfied in $(\mathbf{Q}, <^{\mathbf{Q}})$ and not satisfied in $(\mathbf{Z}, <^{\mathbf{Z}})$. Justify your solutions.

We assume that $R^{\mathbf{N}}$ is $<^{\mathbf{N}}$, $R^{\mathbf{Z}}$ is $<^{\mathbf{Z}}$ and $R^{\mathbf{Q}}$ is $<^{\mathbf{Q}}$.

(4) (20pt) Recall that a partially ordered set (poset) is a pair (A, Q) such that A is a nonempty set and Q is a reflexive, transitive, and antisymmetric relation on A. Let $S = \{R\}$.

(a) Write an S-sentence φ such that $\mathfrak{A} \models \varphi$ iff $\mathfrak{A}$ is a partially ordered set.

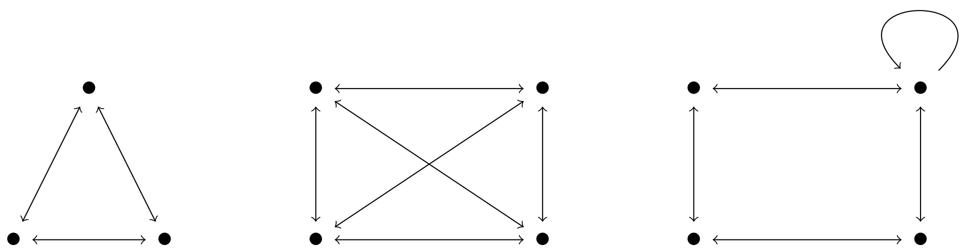
(b) Think of Q as $\leq$. A point $a \in A$ is called *minimal* if there is no b different from a such that Qba. A point a is called the *minimum* if we have Qab for each $b \in A$. Let $S = \{R\}$. Let $\mathfrak{A}$ be an S-structure. Write an S-sentence

(i) $\varphi_{minimal}$ such that $\mathfrak{A} \models \varphi_{minimal}$ iff $\mathfrak{A}$ has a minimal element,

(ii) $\varphi_{minimum}$ such that $\mathfrak{A} \models \varphi_{minimum}$ iff $\mathfrak{A}$ has a minimum.

(c) Show that $\varphi_{minimal} \not\models \varphi_{minimum}$.

(5) (20pt) Let $S = \{R\}$ with R a binary relation symbol. There are three S-structures drawn below. A line drawn between two points means that these points are $R^{\mathfrak{A}}$-related. For each of these structures, give a sentence that is satisfied in this structure but not satisfied in the two others.



$\mathfrak{A}_1$ $\mathfrak{A}_2$ $\mathfrak{A}_3$

(6) (10pt) Let S be a symbol set, φ, ψ and χ S-formulas. Which of the following statements hold? Give either a proof or a counterexample.

(a) $(\varphi \vee \psi) \models \chi$ if and only if $\varphi \models \chi$ and $\psi \models \chi$
(b) $(\varphi \vee \psi) \models \chi$ if and only if $\varphi \models \chi$ or $\psi \models \chi$