WISKUNDIGE LOGICA (2016-2017)
HOMEWORK 3

- Deadline: March 21 — at the **beginning** of class.
- Grading is from 0 to 100 points.
- Success!

(1) (20pt) Is the theory of dense linear orders axiomatizable by universal sentences? That is, does there exist a set of universal sentences Φ such that $\mathfrak{A} \models \Phi$ if and only if $\mathfrak{A}$ is a dense linear order. Recall that density means that if two points are related then there exists a point between them. (Hint: Use the structure $(\mathbf{Q}, <)$ and the substructure lemma.)

(2) (20pt)

(a) Is the union of two consistent sets consistent? If yes, provide a proof, if not, give a counter-example.

(b) Is the intersection of two consistent sets consistent? If yes, provide a proof, if not, give a counter-example.

(3) (10pt) Let $S = \{0, 1, R\}$. Is there a universal sentence φ such that

(a) $(\mathbf{Z}, 0, 1, <) \models \varphi$ and $(\mathbf{R}, 0, 1, <) \models \varphi$?

(b) $(\mathbf{Z}, 0, 1, <) \models \varphi$ and $(\mathbf{R}, 0, 1, <) \not\models \varphi$?

(c) $(\mathbf{Z}, 0, 1, <) \not\models \varphi$ and $(\mathbf{R}, 0, 1, <) \models \varphi$?

(d) $(\mathbf{Z}, 0, 1, <) \not\models \varphi$ and $(\mathbf{R}, 0, 1, <) \not\models \varphi$?

We assume that $R^{\mathbf{Z}}$ is $<^{\mathbf{Z}}$ and $R^{\mathbf{R}}$ is $<^{\mathbf{R}}$. Moreover, $0^{\mathbf{Z}} = 0^{\mathbf{R}} = 0$ and $1^{\mathbf{Z}} = 1^{\mathbf{R}} = 1$.

(4) (20pt) Let P be a ternary relation symbol and f a binary function symbol. Compute:

(a) $P(v_1, f(v_0, v_2), v_3)\frac{v_1\ v_0\ v_5}{v_0\ v_2\ v_3}$

(b) $\exists v_0 P(v_1, f(v_0, v_1), v_3)\frac{f(v_1,v_0)\ v_2\ v_4}{v_1\quad v_0\ v_3}$

(c) $\exists v_0 P(f(v_0, v_2), v_3, v_4)\frac{f(v_1,v_2)\ v_2\ v_4}{v_0\quad v_1\ v_2}$

(5) (20pt) Decide whether the following rules are correct:

(a) $$\frac{\begin{array}{l}\Gamma, \varphi_1 \vdash \psi_1 \\ \Gamma, \varphi_2 \vdash \psi_2\end{array}}{\Gamma, (\varphi_1 \vee \varphi_2) \vdash (\psi_1 \vee \psi_2)}$$

(b) $$\frac{\begin{array}{l}\Gamma, \varphi_1 \vdash \psi_1 \\ \Gamma, \varphi_2 \vdash \psi_2\end{array}}{\Gamma, (\varphi_1 \vee \varphi_2) \vdash (\psi_1 \wedge \psi_2)}$$

(6) (10pt) Let S be a symbol set, f a unary function symbol which does not belong to S. Further, let x and y be different variables and φ an S-formula. Show that

$$\forall x \exists y \varphi \text{ is satisfiable iff } \forall x \varphi \frac{fx}{y} \text{ is satisfiable.}$$