WISKUNDIGE LOGICA (2016-2017)
HOMEWORK 5

- Deadline: May 8 — at the **beginning** of class.
- Grading is from 0 to 100 points.
- Success!

(1) (25pt) Let $\mathfrak{A} = (A, <^A, +^A, \cdot, 0^A, 1^A)$ be a model of the first-order theory of the real numbers. An element $a \in A$ is called an *infinitesimal* if $a > 0^A$ and for each $n \in \mathbf{N}$ we have $a <^A (\frac{1}{n})^A$. (We assume that $(\frac{1}{n})^A \in A$ is an element such that $\underbrace{(\frac{1}{n})^A + \cdots + (\frac{1}{n})^A}_{n-times} = 1^A$.)

Show that there exists a (non-standard) model of the first-order theory of the real numbers that contains infinitesimals. (Hint: use the Compactness Theorem.)

(2) (25pt) Let $\mathfrak{A} = (A, R^A)$ be a structure where R^A is a binary relation. $\mathfrak{A}$ is called *well-founded* if it does not contain an infinite descending chain, that is, there are no elements x_0, x_1, $x_2, \ldots$, such that

$$\ldots R^A x_2 R^A x_1 R^A x_0$$

Is the class of well-founded relations Δ-elementary?

Justify your solution.

(3) (50pt) A set $\Sigma \subseteq \mathcal{P}(I)$ has the *Finite Intersection Property* (FIP) if for any $n \in \mathbb{N}$, from $U_1, \ldots, U_n \in \Sigma$ it follows that $U_1 \cap \cdots \cap U_n \neq \emptyset$.

(a) Show that for every set Σ with the FIP there is a proper filter F (i.e., $F \neq I$) such that $\Sigma \subseteq F$.

(b) Show that for every set Σ with the FIP there is an ultrafilter F such that $\Sigma \subseteq F$. (You can assume the tutorial exercises.)

(c) A proper filter F is called *maximal* if for any other proper filter F' from $F \subseteq F'$ it follows that $F = F'$. Show that F is maximal iff F is an ultrafilter.